Appendix 1

Descartes, Mydorge and Beeckman: The Evolution of Cartesian Lens Theory 1627-1637

1. Introduction:

In Chapter 4 Section 5.2 it was foreshadowed that the analysis of the trajectory of Descartes' lens theory can provide crucial supporting evidence for my reconstruction of Descartes' discovery of the law and his attempts at its physico-mathematical rationalisation, and that in particular, it can do this by allowing us quite firmly to date the material in the Mydorge letter to the period 1626/7 shortly after the law of refraction was discovered, in cosecant form, by Descartes and Mydorge.

In this Appendix we are going to canvass the following key points: [1] In constructing his lens theory Mydorge begins with the cosecant form of the law and only finds a sine formulation in the course of elaborating the theory. [2] His synthetic proofs of the anaclastic properties of plano-hyperbolic and sphero-elliptical lenses are similar to, but clearly pre-date those offered by Descartes later in the *Dioptrique* of 1637. Moreover, [3] Descartes own synthetic lens theory demonstrations in the *Dioptrique* differ from those of Mydorge in another historically revealing way, the matter turning on a technical and aesthetic issue which Descartes seems to have learned from Beeckman in October 1628. In other words Descartes' lens theory developed during three moments between 1626/7 and the publication of the *Dioptrique*: First, we have the earliest lens theory of Descartes and Mydorge in the Mydorge letter, whose content dates from 1626/27; second, we shall see some consequential shifts and articulations in Descartes' theory as a result of consultations and negotiations with Isaac Beeckman in 1628; and, finally, we have the synthetic lens theory of the *Dioptrique* of 1637.

All these facts will therefore suggest that the Mydorge letter contains Mydorge and Descartes' *earliest lens theory*, and arguably *their first form of the law*, the cosecant form. The *material in the letter,* if not the artefact itself, pre-dates October 1628, certainly predates composition of the *Dioptrique* and very plausibly is as early as 1626/7. So, this dating points to the cosecant form of the law as the first form Mydorge and Descartes possessed. This, as we have seen, is the key to reconstructing how they obtained it, because the other independent discovers first obtained it in the same *unequal radius form*.[1] We start therefore by returning the Mydorge letter, intending to analyse all those parts of it not examined in our earlier discussion in Chapter 4. Having

[1] Lohne (1959), (1963); Vollgraff (1913), (1936); deWaard (1935-36); Buchdahl (1972).

already looked only at Mydorge's Proposition 1, his statement and geometrical illustration of the cosecant law of refraction, we begin with his second proposition:

2. Mydorge's Refractive Index Instrument: Cosecants not Sines

In Proposition 2 of his letter, Mydorge explains a device used to determine the refractive index of a given medium, in this case the glass Descartes and Mydorge apparently intended to use in the fabrication of lenses (**Figure Ap1.1**). Mydorge sends a ray, FG, through the triangular prism of glass ABC. The ray enters the prism normal to AB and is refracted at AC to E. DIH is the normal to AC at I.[2] The geometry of the device is elegant. The angle of incidence FID is equal to the angle BAC and hence is known in advance. The angle of refraction HIE is equal to the sum of angles FID and IEC.

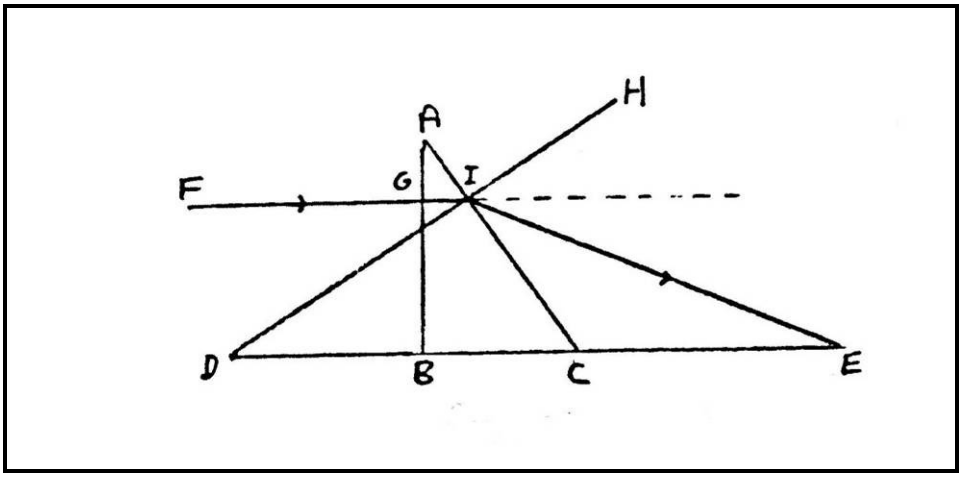

Figure Ap1.1

Only one measurement, that of angle IEC need be made in order to determine the refractive index, for

$$\frac{\sin \text{ FID}}{\sin (\text{FID} + \text{IEC})} = \text{Index refraction}$$

Curiously, however, Mydorge does not exploit the device in this manner, by taking the sines of the angles of incidence and refraction. Instead, he relies upon the radius form of the law taught in Proposition 1 (**Figure Ap1.2**). Around I he draws the arc of the circle of radius FI. He

[2] Mersenne (1932-88) I. p.405

constructs FK parallel to AC cutting the arc FK at K; then from K he drops a line parallel to DIH cutting the refracted ray IE at L. The ratio IL:FI is the index sought.[3]

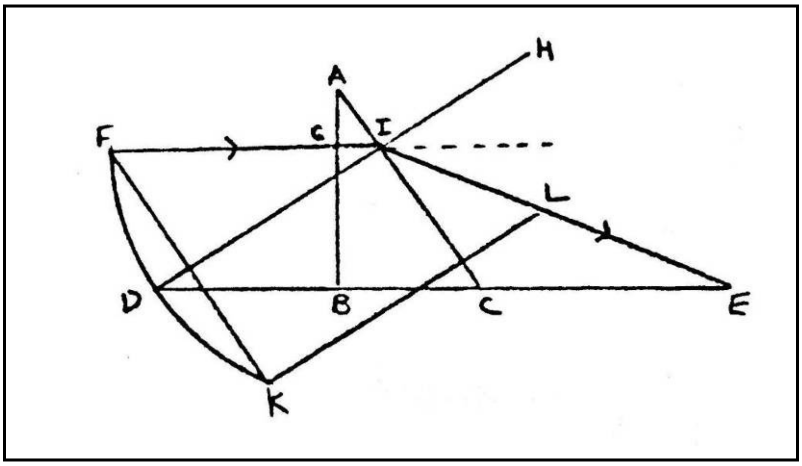

Figure Ap1.2

Now, had Mydorge discovered the law in what we have called sine form, he could have greatly simplified the whole discussion and diagram. A likely interpretation is that the radius form of the law was prior to the sine form. In seeking to exploit the law in an experimental situation, Mydorge apparently reached for the only form of it with which he was acquainted, the radius form. There is further evidence for this interpretation in the remainder of the letter.

3. Mydorge's Synthetic Propositions 3 and 4 on Anaclastic Surfaces: An 'Antique' Version of the Sine Law

Propositions 3 and 4 of Mydorge's are devoted to lens theory proper.[4] Applying the law of refraction to an hyperbola (prop. 3) and to an ellipse (prop. 4), Mydorge shows that if an incident ray, parallel to the transverse axis of either of these conics, is refracted at its point of incidence with the section to the appropriate focus, then,

$$\frac{\sin i}{\sin r} = \frac{\text{transverse axis}}{\text{focal distance}}$$

The case of the hyperbola is illustrated in **Figure Ap1.3**. CBA is the left branch of the hyperbola, D and E its foci, BF the transverse axis, CH the tangent to the section at C, and CI the normal to the section at C. The incident ray GC is refracted at C to the distant focus D. Mydorge uses the sine form of the law of refraction, representing the sine of the angle of

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[3] *Ibid*. pp. 406-7. That is, the constant ratio IL:FI and the construction technique used in Proposition 1 will yield the paths of all other refracted rays. Cf. Above Chapter 4, Section 5.1 and Figure 4.5.

[4] *Ibid*. pp. 408-9

incidence GCI by GL and the sine of the angle of refraction ICK by KM. We should note especially here for later reference the odd, or as I shall call it 'antique', representation of the sine law, with the sines of incidence and refraction inscribed on the same side of the refracting surface.

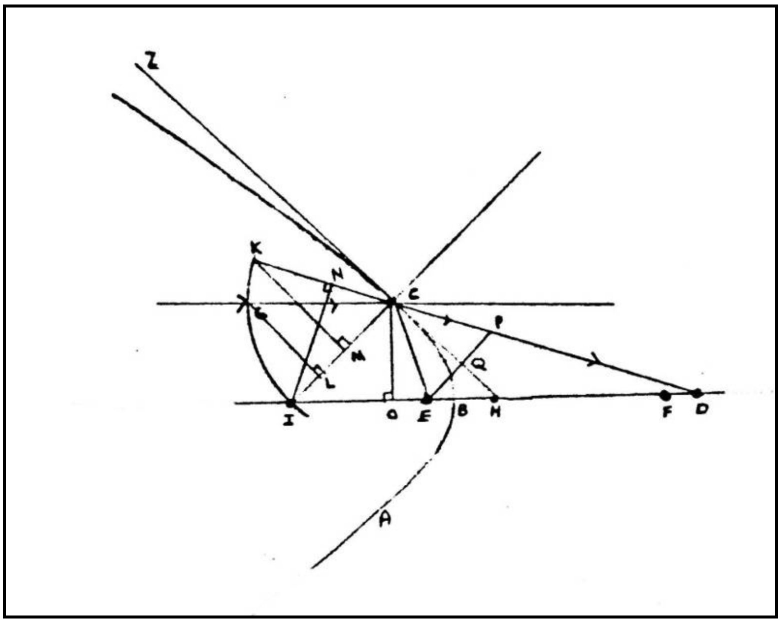

Figure Ap1.3

Mydorge easily demonstrates that (GL:KM)=(BF:DE), the proportion sought between the ratio of the sine of the angle of incidence to the sine of the angle of refraction, and the ratio of the transverse axis to focal distance of the hyperbola.[5] Since ray GC and point C were selected at random, the demonstration applies to any such ray parallel to the transverse axis and refracted by the section to the distant focus. The relevance to lens theory is clear, although it is not spelled out by Mydorge in these propositions: If an hyperbola, defined by the ratio of transverse axis to focal distance of BF:DE, were embodied in a convex plano-hyperbolic lens made of a transparent material the index of refraction from which into air were equal to BF:DE, that lens would focus all rays incident parallel to its transverse axis to its distant focus—it would embody an anaclastic surface.

Mydorge's use of a sine form of the law of refraction, albeit in the unusual 'antique' form, might seem to undermine the claim that he and Descartes first discovered the law in radius form. This, however, is not the case, as we can see by placing Propositions 3 and 4 in the two relevant contexts which facilitate their accurate interpretation. The first of these contexts is the

[5] The proof proceeds easily and in routine fashion based on well known properties of the conics, chiefly by means of deduction through a sequence of equal and similar triangles inscribed in the figure. The proof tactics in these routine *concluding stages* are identical to those Mydorge uses in his Proposition 5 discussed below, and in Descartes' corresponding proof for the plano-hyperbolic lens in the *Dioptrique.* Here we are concentrating on *opening stages* of these proofs, where various representations of the law of refraction are adduced and further manipulated.

subsequent history of Propositions 3 and 4 down to their publication by Descartes in the *Dioptrique* of 1637. We shall see in the next Section that Mydorge's proofs sit at the very beginning of this history, during which the 'antique' version of the sine law was transformed into our familiar, let us say 'natural' form, in which the sines of incidence and refraction are assigned to their respective sides of the refracting interface. All this will strongly reinforce our earlier conjecture that the material in the Mydorge letter dates from 1626/27, the very period of the initial discovery of the law of refraction. With that conclusion in hand, we will then turn in Section 5 to the second context of Propositions 3 and 4, which is the surrounding text of Mydorge's letter itself; that is, Propositions 1 and 2, which we have discussed, and Proposition 5, his final proposition, which we will have to examine with great care. Proposition 5 shows how Mydorge connected the putatively original, radius or cosecant form of the law to a sine form of the law, but only in its first or 'antique' version. Additionally, because this material is quite early, we will be able to detect in Proposition 5 echoes of Mydorge's (and Descartes') earliest analysis of the anaclastic problem, the very beginning of their research on lens theory with a law of refraction in hand. *We will conclude in Section 6 that the 'antique' sine form was evolved out of the radius form of the law during the course of this analysis.* In other words, we shall see that the sine law in its initial, 'antique' form was discovered during the course of an analysis of the anaclastic problem initially launched on the basis of the newly discovered radius form of the law. The 'antique' sine form, after having been uncovered in this way, was then deployed in the more synthetic Propositions 3 and 4, with Descartes' more 'natural' representation of the law, only unveiled in the *Dioptrique* nowhere in sight (until Beeckman suggested it in October 1628).

4. Relating Mydorge's Propositions 3 and 4 to Descartes' Analogues in the *Dioptrique*: From 'Antique' to 'Natural' Representation of the Sines, Thanks to Isaac Beeckman in October 1628

First let us consider the place of Mydorge's propositions 3 and 4 in the development of Cartesian lens theory between 1627 and 1637. In the *Dioptrique* Descartes proves propositions identical to those of Mydorge, but they differ in one historically revealing way. Instead of setting up the sines of the angles of incidence and refraction by reference to a semi-circle on one side of the interface, as Mydorge had done, Descartes directly relates the sines to their respective rays. Consider **Figures Ap1.3 and Ap1.4** where this point is illustrated using Mydorge's and Descartes' figures for the case of the hyperbola. (Similar considerations would apply in the cases of their figures for the ellipse).

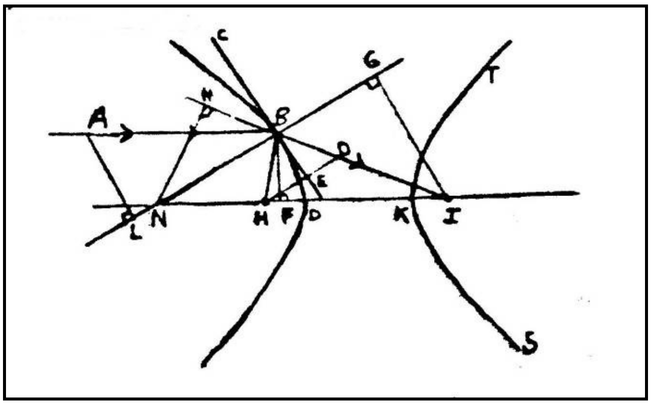

Figure Ap1.4: Descartes' Hyperbola Proof in the *Dioptrique*

In Descartes' diagram **(Figure Ap1.4)** H and I are the foci of the hyperbola, D and K its vertices; ray AB is refracted at B to I; CBE is tangent to the left branch at B; LNBG is normal to CBE at B; BA is taken equal to BI so that AL and GI represent respectively the sine of the angle of incidence and the sine of the angle of refraction in the 'natural' representation of the law, familiar since that time. The proof that the ratio of the sines of the angles of incidence and refraction [AL:GI] equals the ratio of the transverse axis to the focal distance [DK:HI] again follows easily on routine knowledge of the conics, utilising a sequence of relations amongst equal and similar triangles in the figure.[6] In Mydorge's diagram **(Figure Ap1.3)**, we recall GL and KM are the sines of the angle of incidence and the angle of refraction respectively, given in 'antique' form as we have already seen.

Now, we can actually pin down the likely source of Descartes' later 'natural' representation. Descartes' friend Isaac Beeckman seems to have been the author of Descartes' mature representation of the sines in the context of lens theory. In 1628 Descartes asked Beeckman to provide a proof of the refractive properties Descartes had claimed for the hyperbola. Beeckman's proof omits several steps and does not fully specify the construction. But geometrically it is identical to **Figure Ap1.4** and was 'approved' by Descartes.[7] At the same time, in 1628, Descartes showed to Beeckman an elegant proof for the case of the ellipse.[8] However, he did not subsequently use that proof in the *Dioptrique*, probably because the lines representing the sines of incidence and refraction are not related to their respective rays in the intuitively obvious way displayed in **Figure Ap1.4**. One can conclude that Descartes elected to use Beeckman's more 'natural' representation of the sines in both cases, ellipse and hyperbola, in the synthetic proofs in the *Dioptrique*, thus superseding his own elegant ellipse proof and Mydorge's early 'one sided' representation of the sines in Propositions 3 and 4. This episode

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6 *Dioptrique*, AT VI p. 178.

7 AT x. 341-2; Beeckman, *Journal* fol. 338r

8 *Ibid.*

with Beeckman, which we may imagine to have been in the nature of a negotiation (and set of mutual challenges, as befitted their previous interactions in 1618-1619) marks the second moment in the evolution of Descartes' lens theory.

To sum up so far: The development of the lens theory proofs places Mydorge's Propositions 3 and 4 very early in his and Descartes' researches. In terms of proof content and diagrammatic representation, Propositions 3 and 4 are the earliest proofs in their lens theory of which we have any record; and they are clearly the starting point for Beeckman's and Descartes' later improvements. Mydorge's demonstrations obviously pre-date Descartes' and Beeckman's discussions of lens theory in 1628, and hence they arguably date from the very period of the discovery of the law of refraction. This, accordingly, aids in our dating of all of the material in the Mydorge letter from 1626/27. The dating becomes even more likely when one considers that by 1632 the Cartesian sine form of the law was well known to several of Descartes' associates, including Golius and Mersenne, in addition to Beeckman. In informing Golius about his optical work Descartes mentioned only the sine from of the law.[9] But in his letter Mydorge, Descartes' closest associate, does not initially use the sine form, and when he does introduce it, in his lens theory, he produces an early 'one-sided' version soon superseded in Beeckman's and Descartes' proofs. It is therefore most unlikely that the material in the letter was initially composed in 1631 or later, the possibility left open by De Waard when he tried to date the letter. All the evidence points toward the conclusion that the material in the Mydorge letter was an early and rather unsystematic and undigested report on his and Descartes' researches of 1626/27.

5 Decoding Mydorge's Proposition 5: The cosecant form leads to 'discovery' of the 'antique' sine form then used synthetically in Propositions 3 and 4.

With these conclusions about the early date of the letter in mind, we can now proceed to the second context of Propositions 3 and 4, the surrounding text of the letter, and in particular Proposition 5. What we are after is an explanation of Mydorge's use of the sine form of the law in Proposition 3 and 4, an explanation grounded in an understanding of the surrounding portions of Mydorge's text and framed by our now strong conviction that the material in the letter is

[9] Descartes to Golius, 2 February 1632, AT I. p.239ff. When Descartes met Beeckman in October 1628 he offered him a striking and very important mechanical analogy for the law-like refraction of light, appealing to a bent arm balance supporting identical weights, whose arms are immersed in media (upper and lower) of differing specific gravities. Chapter 4, Section 7.4 above and Schuster (2000) pp. 290-295, show how this analogy directly bespeaks Descartes' dynamical thinking about the absolute force of light and its determinations, before and after refraction. The bent arm balance, however, is presented to Beeckman using representations of the sines of the incidence and refraction of the arms, on their respective sides of the interface—the 'natural' representation of the sine law we are talking about. The issue is that whilst Descartes had by 1628 worked out this model for his dynamics of light, the representation of the sine law embodied in it was not applied back into lens theory until Beeckman suggested it. Presumably, Descartes still had to hand proofs resembling those of Mydorge from 1626/27.

indeed of very early provenance, dating back to the period of the discovery of the law of refraction.

Proposition 5 deals with the specification of hyperbolic and elliptical anaclastic curves in actual empirical cases. It amounts to a linking of Proposition 2 with Propositions 3 and 4. Mydorge starts by showing how to measure the index of refraction for rays passing into the air from the glass out of which the lenses are to be fashioned. Exactly as in Proposition 2, the index is determined by passing one ray through a triangular glass prism, and the index is expressed as a ratio of radii (not as a ratio of sines) by applying the radius form of the law of refraction to the given ray. That is, in **Figure Ap1.5**, which shows the first few steps in Mydorge's fifth proposition, the glass:air index is given as IL:FI.

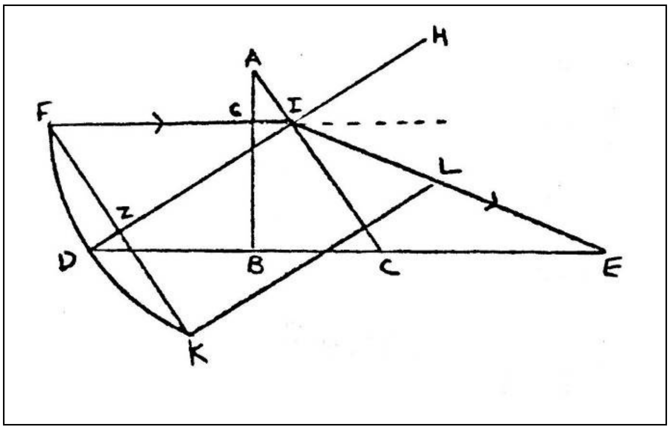

Figure Ap.1. 5

Next, the empirically determined index is used to set the ratio of transverse axis: focal distance for the hyperbolas and ellipses in question. This construction is effected by exactly repeating a construction Mydorge had already given as a Corollary to Proposition 2.[10] Taking the case of the hyperbola only, **Figure Ap1.6** shows how this construction is added to the material previously assembled in **Figure Ap1. 5**.

[10] Mersenne (1932-88) I pp. 406-7; In the *Dioptrique* (AT VI pp. 212-3) Descartes recapitulates the material in Mydorge’s Proposition 2: He presents the same refraction device and then shows how to interpolate the transverse axis and foci of the anaclastic hyperbola into its geometry.

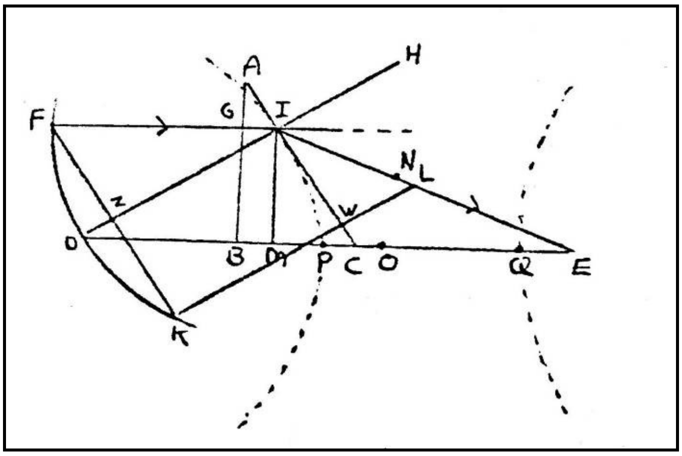

Figure Ap1.6

AC, representing the refracting surface of the prism, is considered to be tangent to the left branch of the hyperbola at I, the point of incidence. IE, the refracted ray, is taken to have passed through the distant focus at E. The near focus is then located by using the property of the hyperbola that a tangent to a point in the section bisects the angle between the lines drawn from that point to the two foci.[11] Thus AC bisects angle EIM and M is the left focus. IN (=IM) is marked off along IE. Then, according to the basic property of the hyperbola, the difference between IE and IN gives the length of the transverse axis, PQ, which can easily be inserted between the two foci. Finally, Mydorge must show that as IL:FI so PQ (or NE):ME. With the exception of its first step this proof is identical to that given in Proposition 3. The only difference is that Proposition 3 begins with a sine form of the law, whilst here we must start the proof with the radius form of the law, the product of Mydorge's introduction of the index measuring technique of Proposition 2.

The crucial step linking the radius form, IL:FI, to the sine form consists in a simple construction, which we add to Figure Ap1.6 in **Figure Ap1.7**.

[11] Mydorge refers the reader to the relevant proposition in his own work on *Conics* first published in 1631; but he may of course be referring to draft material prior to that date. We can insist on dating the content of the Mydorge letter from 1631 or later, based on the publication history of his *Conics* and his use of routine material from it; or we can look closely at the absolutely novel aspects of the letter—the adducing of the cosecant form of the law and the working through from it to the 'antique' version of the sine law— placing all of that material in the context of whatever else we can reconstruct about Descartes' work on explaining the law of refraction and on lens theory. To reiterate, the latter considerations, argued here and in Schuster (2000), conduce to the hypothesis that the *material* in the letter dates from the very period of discovery of the law of refraction, 1626/27, and not from the early 1630s when mature versions of all this material, well beyond the toing and froing of Mydorge's letter were known in the Descartes/Mersenne network, and Mydorge's presentation would have seen oddly out of date and out of touch.

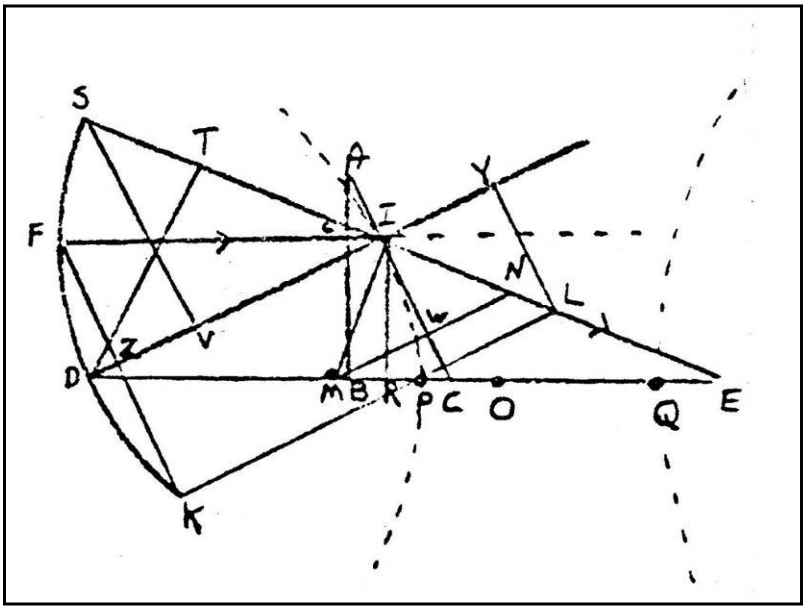

Figure Ap1. 7

Arc KDF is extended about point I, and ray IE is extended back to intersect the arc at S. SV and LY are dropped perpendicular to DI. Then FZ (=LY) and SV are the sines of the angles of incidence and refraction respectively, referred to the single reference circle KDFS. Triangle ISV is similar to triangle ILY, hence (SI:IL) = (SV:YL); but SI = FI and YL=KZ=FZ, hence (IL:FI) = (FZ:SV). After dropping DT normal to SIE and IR normal to ME, Mydorge can then continue the proof in the manner of Proposition 3 by making use of the equal triangles FIZ and IRD, and DTI and SVI; and the similar triangles DTE and IRE, and DIE and MNE. [12]

Proposition 5, as Mydorge presents it, marks the final step in his unfolding of his lens theory. First, using the radius form of the law only, he expounded the law of refraction (Proposition 1) and his index finding technique (Proposition 2). Then, using his 'antique' sine form of the law, he offered demonstrations linking the law of refraction to the ratio 'transverse axis:focal distance' of hyperbolas (Proposition 3) and ellipses (Proposition 4). In proposition 5 he turns to empirical cases of plano-hyperbolic and sphero-elliptical lenses, which means he must connect Proposition 2 to Propositions 3 and 4. This explains, in the context of the letter, Mydorge's construction of the 'antique' sine form of the law out of the radius form, and his redundant repetition of the proof structure of Propositions 3 and 4 in Proposition 5.

Of course, Mydorge's order of presentation of lens theory need not have corresponded to his and Descartes' order of research and discovery in this domain. If, as seems very likely, he initially had only the radius form of the law, he could not have pursued his synthetic lens theory through Propositions 3 and 4 without first having uncovered, by analysis, a way of linking the radius form to the fundamental properties of the conic sections. This way of linking was his

[12] Cf above Note 5

construction of the 'antique' sine form of the law out of the radius form. If one asks what Mydorge's (and Descartes') path of analysis might have looked like, a very plausible candidate springs to view—Proposition 5 itself. In the context of the letter the bulk of the proof of Proposition 5 is redundant and repetitive; but, if Proposition 5 is read, as it were, backwards, as a remnant of an analysis, we obtain a story about Mydorge and Descartes' possible original analysis of the anaclastic problem which, given everything which has gone before, seems very plausible indeed.

6 A Reconstruction of Descartes and Mydorge's First Analysis of the Anaclastic Problem, with the Cosecant Law of Refraction to Hand.

Let us therefore try to reconstruct the analysis of which Proposition 5 seems to contain the remnants, and let us do this on the basis of the relevant facts we have already more or less established about Mydorge and Descartes' early optical work and intentions. First of all, as established in Chapter 4 above, we must imagine that Mydorge (and Descartes) obtained the radius form of the law by means of an image mapping technique similar to that used by Harriot. Next, we must hypothesise that with the radius form in hand, they moved to explore the possibility, hinted at by Kepler, that the hyperbola might be the anaclastic curve.[13] Drawing upon their combined knowledge of the conic sections, they would have designed their elegant experimental device (if only on paper at first!) in such a way that they could easily interpolate an hyperbola whose defining property would be entailed by the geometry of the prism and the behaviour of the empirically given ray, incident parallel to the intended transverse axis of the hyperbola. Then they would have had to attempt to prove the relation of the cosecant regularity to some expression of the defining property of the interpolated hyperbola, thus showing that the refraction to the distant focus holds for any parallel incident ray, and hence that the left branch of the hyperbola is an anaclastic surface. In the manner of classical geometry, and in accord with Descartes' explicit views on mathematical method, the analysis could then be reversed in so far as possible to guide the production of synthetic propositions such as Mydorge's Propositions 3 and 4.

With this background, the anaclastic problem would have taken the following form: Assume an incident ray parallel to the transverse axis of the hyperbola is refracted by the section to the distant focus. Can the law of refraction (reflecting the index of refraction) be related to the ratio 'transverse axis:focal distance' characterising the hyperbola in question? So, let us imagine in **Figure Ap1.8** what Mydorge and Descartes' analysis diagram might have looked like: Assume we are given hyperbola IPW, with foci M and E, and ray FI refracted at I to E. IC is tangent to the hyperbola at the point of incidence I. As accomplished geometers and experts on the conic

[13] Kepler, *Dioptrice* (1611). Prop 59.

sections we also know that angle MIC=angle CIE; that IM=IN and NE=PQ = transverse axis. Next we construct the index of refraction in the only form we know, in radius form, as the ratio IL:FI.

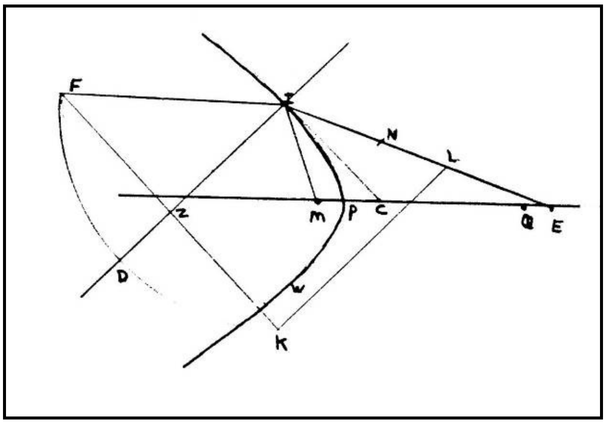

Figure Ap1.8

Before proceeding any further, we need to make two observations. First, point D has not been located on the extension of ME, as in Proposition 5 **(Figures Ap1.5, Ap1.6 and Ap1.7)**, because the exact location of D was probably discovered in the course of the analysis. Note also that **figure 8** has been constructed solely on the basis of (1) knowledge of the radius form of the law, (2) the suspicion that the hyperbola is the anaclastic curve, and (3) elementary knowledge of the properties of hyperbolas, which was second nature to Mydorge and Descartes.

Given **Figure Ap1.8** the analytical problem is to show that (IL:FI) = (PQ:ME). Mydorge's synthetic demonstration of this relation in Proposition 5 relied on the establishment of the sine form of the law (FZ:SV) and on the relating of (FZ:SV) to (PQ:ME) via the set of equal and similar triangles pointed out earlier in **Figure Ap1.7**. Clearly, the successful analysis of the problem set in **Figure Ap1.8**, by Mydorge, Descartes or any one else, demands two crucial steps: One has to discover that point D must be located on the extension of EM, otherwise the series of interrelated equal or similar triangles does not materialise; and, one must also transform IL:FI into some ratio of lines relatable to the limbs of those triangles. *This is the very trick Mydorge accomplishes by constructing the 'one-sided' sine form of the law, FZ:SV in **FigureAp1.7***. Precisely how Mydorge and Descartes made those moves and in what order, we cannot know. That they performed an analysis of this general type is highly likely, given the close relation between Mydorge’s **Figure Ap1.7** and our **Figure Ap1.8**, constructed in light of what we can determine about the direction, background and tools of their early researches. This sort of analysis invites the construction of Mydorge’s peculiar 'one-sided' version of the sine form, and such a route to Proposition 5 would explain Propositions 3 and 4 as later, synthetic versions of this material, launched, for simplicity's sake, on the basis of the ‘antique’ one–sided sine form. To put the matter quite generally, if you initially have only the radius form of the law of refraction and are analysing the anaclastic properties of hyperbolas using the resources of classical and renaissance geometry, then you are very likely to construct the ‘antique’ sine form of the law in order to consummate the analysis. In this situation the 'one sided' form of the sine law is particularly useful and likely to turn up.

This necessarily technical section can be brought to a close by summarising in 'synthetic' fashion the main conclusions we have reached through our complicated 'analysis' of the Mydorge letter, stage one of Descartes’ lens theory. Below in drawing larger conclusions, we shall work in observations involving Stage two, the interaction with Beeckman in 1628, and Stage 3, the form of the theory published in the *Dioptrique* of 1637.

(1) The evolution of Mydorge and Descartes' lens theory shows that the content of the Mydorge letter dates from before 1628 and therefore approximates to the date of the discovery of the law of refraction in 1626/27.

(2) Given (1), Mydorge's initial reliance upon the radius form of the law in his propositions 1 and 2 most likely indicates that this was the first form of the law with which he was acquainted, presumable because he and Descartes discovered the law through the 'Harriot-like' procedure of mapping image places using the traditional image placement rule, and using data which need have been no better than those supplied by Witelo.

(3) Given (1) and (2), Mydorge's Proposition 5 can be read as containing remnants of Mydorge and Descartes' initial analytical investigations of lens theory, using the radius form of the law as a tool. This path of analysis turned up the ‘antique’ sine form of the law which was then used in devising the proofs of Propositions 3 and 4.

7 The Kramer-Milhaud Thesis: Discovering the Law of Refraction by Analysis of the Anaclastic Problem

Our results to this point permit us to evaluate a conjecture concerning the genesis of Descartes' law of refraction which has commanded a fair degree of credence for over a century. P. Kramer in 1882, followed by Gaston Milhaud in 1907, suggested that Descartes discovered the law of refraction as a result of posing and analysing the anaclastic problem in this form:

> Given an ellipse or hyperbola, on which a ray falls parallel to the focal axis, according to what geometrical condition will the ray be refracted to one of the foci?[14]

[14] Kramer (1882); Milhaud (1907)

Taking the case of the ellipse, Milhaud sketched an analysis beginning in the following fashion **(Figure Ap1.9)**:

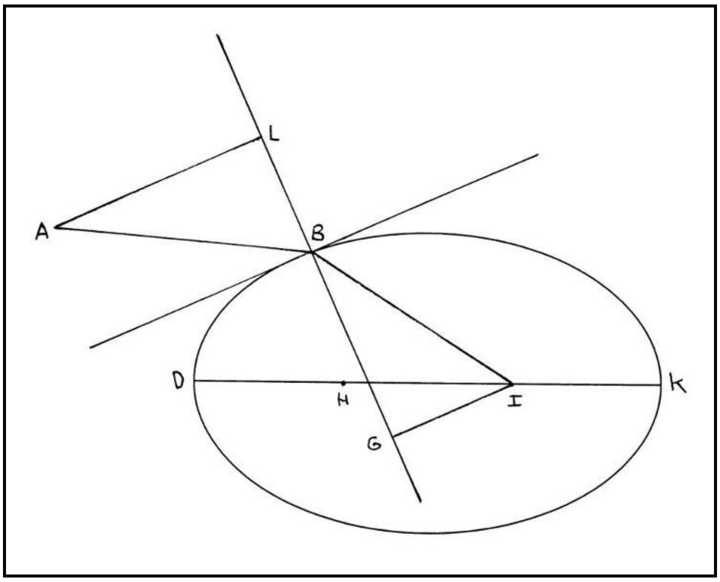

Figure Ap1.9

Ray AB enters the ellipse at B parallel to axis DK and is refracted to focus I. Lay off BA = BI and drop a normal LB to the tangent to the ellipse at point B. Then to this normal drop the sines of the angles of incidence and refraction, AL and IG respectively. Milhaud correctly stated that it can easily be shown that

$$\frac{AL}{IG} = \frac{DK}{HI} \qquad \text{Or,}$$

$$\frac{\sin i}{\sin r} = \frac{\text{transverse axis}}{\text{focal distance}}$$

and that the young Descartes could easily have done this.[15]

Milhaud did not give the rest of the analysis; but it would chiefly consist in a reversal of the steps found in Descartes' demonstration in the *Dioptrique* of the equivalent of Mydorge's Proposition 4. For Kramer and Milhaud this route of discovery had the virtue of exploiting Descartes' mathematical expertise whilst eliminating any appeal to experiment. (In addition one can point to the existence of a synthesis of the problem published by Descartes himself.)

Kramer and Milhaud were perfectly correct to believe the law could have been discovered in this fashion. Indeed, their conjecture could have been made even more plausible had they

[15] Milhaud (1907), p. 226.

defined less tendentiously the analytical problem they attribute to Descartes. There was no need to specify Descartes' (really Beeckman's) version of the sine form of the law as the fruit of the analysis. Mydorge's one sided version would have served equally well, as would any equivalent construction, for example, the neat construction used in Descartes' elegant but suppressed 1628 proof for the case of the ellipse, mentioned above. Strictly, speaking, moreover, Kramer and Milhaud need not have specified any sine form of the law (or indeed any form of the law at all) in the data of the problem. It would have been more historically plausible simply to posit Descartes beginning with an ellipse or hyperbola and a parallel incident ray refracted to the distant focus. In such conditions, anyone with a knowledge of the conic sections could have easily discovered the relation between the ratio of the sines of incidence and refraction and the ratio of transverse axis to focal distance, by identifying the angles equal to the angles of incidence and refraction (or to their complements or supplements) and by applying the trigonometric law of sines.

Unfortunately, however, with or without such improvements, the Kramer-Milhaud conjecture suffers from one serious weakness: there is no positive evidence for it, and the evidence which can be teased out of the Mydorge letter runs directly counter to it. There is no evidence in Mydorge's letter of the law of refraction having been discovered by a straightforward analysis of the anaclastic problem. Mydorge's proofs are loaded with the one sided sine form and/or the radius form of the law; they are hardly the results of the elegant analysis envisioned in the Kramer-Milhaud thesis. Significantly, neither Harriot nor Snel give any evidence of having performed an analysis of that type. In addition, the evidence in Proposition 5 of Mydorge and Descartes' early analytical work in lens theory suggests that their analysis began with the radius form to hand. Their problem was to relate the radius form to the defining properties of an hyperbola or ellipse, operating on the not entirely wild suggestion of the authoritative Kepler that these conics could provide anaclastic surfaces. Descartes and Mydorge had the law, in cosecant form, already to hand, and needed to explore whether it could be related to the defining properties of the conics. Kramer and Milhaud require an initially entirely theoretical and mathematical procedure, producing a *candidate* law of refraction (in some trigonometric form or other as noted above), which then would have had to have been explored from an empirical point of view. However, it should be obvious from the total content of the Mydorge letter, properly interpreted, that the probability of the Kramer-Milhaud thesis being historically accurate is virtually nil.

8. Conclusions

[1] The reconstruction of the evolution of Descartes' lens theory confirms my claim in Chapter 4 that Mydorge and Descartes first stumbled on the law of refraction in cosecant rather than

sine form, because it establishes that the sine form of the law only emerged during the course of analysis of the anaclastic problem, once the cosecant form was in hand.

[2] The reconstruction shows that the sine form provided more elegant lens theory propositions than the cosecant form, hence motivating its *overall* use in the *Dioptrique*, *even* in the problematical and confusing 'proof' of the law of refraction.

[3] This in turn further explains my finding in Chapter 4 that the 'natural' version of the sine law, when used in relation to the 'tennis ball model of light' proof of the law of refraction in the *Dioptrique,* created problems of exposition and understanding that would not have occurred had Descartes used the cosecant form, and a more explicit version of his 'dynamics of light (which on my reconstruction was derived from a physico-mathematical reading of the cosecant form).

[4] Similarly, the reconstruction shows that Beeckman introduced Descartes to the 'natural' form of the sine law for use in lens theory, whilst Descartes showed him an even more elegant representation for lens theory proof purposes. Because of its utility both in lens theory and in setting out the tennis ball model for the action of light in the attempted derivations of the optical laws, Descartes ultimately opted for the former over the latter. However, the original cosecant form of the law, which, as we have discovered, more accurately modelled the dynamical concepts underlying the optical proofs—having itself inspired their formulation—was never used by Descartes in geometrical representations of the law of refraction or in its supposed proof, although some of his verbal formulations in answers to critics of the *Dioptrique*, betray just that underlying conceptualisation.[16]

[5] My argument shows that Kramer and then Milhaud's reconstruction of Descartes' path of discovery of the law, which invoked a process of *de novo* and completely mathematically abstract analysis of the anaclastic problem, cannot be correct, given the documentary evidence available. However, it is fair to say that the Kramer/Milhaud conjecture was, as far as it goes, consistent with my claim that *the sine law did indeed emerge in the course of an analysis of the anaclastic problem*, provided, however, *Descartes and Mydorge already possessed and deployed in that analysis the cosecant form of the law*, itself having been obtained through other, quite different, and quite traditional mixed mathematical manoeuvres in geometrical optics.

[6] In general, then, the sine form of the law emerged within, and became elegantly functional to, the development of lens theory, given the prior existence of the cosecant form of the law. By contrast, as we learned in Chapter 4, the original cosecant version of the law was intimately connected with Descartes' attempt to derive physico-mathematical capital from geometrical

[16] See for example Descartes' remarks to Mydorge for Fermat in March 1638, Chapter 4 Note 25 and Schuster (2000) Note 24.

optics, first by reading out dynamical principles governing the behaviour of light, and thence by promoting those principles to the level of a general dynamics of corpuscles.